Assignment 11

This homework is due Monday April 20.

There are total 50 points in this assignment. 45 points is considered 100%. If you go over 45 points, you will get over 100% for this homework (but not over 115%) and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper *and give credit to your collaborators in your pledge.* Your solutions should exhibit your work and contain full proofs. Bare answers will not earn you much.

This assignment covers Sections 6.6–7.2 of Textbook.

Recall that $C_\rho(z_0)$ denotes the circle of radius ρ centered at z_0. $D_\rho(z_0)$ denotes the open disk of radius ρ centered at z_0. $\overline{D}_\rho(z_0)$ denotes the closed disk of radius ρ centered at z_0.

(1) [10pt] Let f be analytic in the disk $D_5(0)$ and suppose that $|f(z)| \leq 10$ for $z \in C_3(1)$.
   (a) Find a bound for $|f^{(4)}(1)|$. (*Hint:* Use Cauchy Inequalities.)
   (b) Find a bound for $|f^{(4)}(0)|$. (*Hint:* Use Cauchy Inequalities for the circle $C_2(0)$. To get a bound on $|f(z)|$ on $C_2(0)$, remember that $C_2(0) \subseteq \overline{D}_3(1)$ and use Maximum Modulus Principle.)

(2) [10pt] Use Weierstrass M-test to show that the following series converge uniformly on the sets indicated.
   (a) $\sum\limits_{k=1}^{\infty} \frac{1}{k^2} z^k$ on $\overline{D}_1(0) = \{z : |z| \leq 1\}$.
   (b) $\sum\limits_{k=0}^{\infty} \frac{1}{(z^2-1)^k}$ on $\{z : |z| \geq 2\}$.
   (c) $\sum\limits_{k=0}^{\infty} \frac{z^k}{z^{2k}+1}$ on $\overline{D}_r(0)$, where $0 < r < 1$.

(3) [10pt] By computing derivatives, find the following Taylor series for the functions below and state where it is valid.
   (a) $\cos z$ at $z = 0$.
   (b) $\cos z$ at $z = \frac{\pi}{2}$.
   (c) $\sinh z$ at $z = 0$.
   (d) $\sinh z$ at $z = \frac{\pi}{2} i$.
   (e) $\operatorname{Log}(1 + z)$ at $z = 0$.
   (f) $\frac{1}{z}$ at $z = 1$.

(4) [10pt] Using methods other than computing derivatives, find the Maclaurin series (=Taylor series at 0) for the following functions.
   (a) $\cos^3 z$. (*Hint:* $4\cos^3 z = \cos 3z + 3\cos z$.)
   (b) $\operatorname{Arctan} z$. (*Hint:* Integrate the Maclaurin series for $\frac{1}{1+z^2}$.)
   (c) $(z^2 + 1)\sin z$. (*Hint:* $(z^2 + 1)\cos z = z^2 \cos z + \cos z$.)
   (d) $f(z) = e^z \cos z$. (*Hint:* Express cos through the exponential, expand the brackets.)

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(5) [5pt] Let $f(z) = \frac{\sin z}{z}$ and set $f(0) = 1$.
  (a) Explain why f is analytic at $z = 0$.
  (b) Find the Maclaurin series for $f(z)$.
  (c) Find the Maclaurin series for $g(z) = \int_C f(\zeta)d\zeta$, where C is a straight line segment from 0 to z.

(6) [5pt] Let $f(z) = (1+z)^\beta = \exp(\beta \operatorname{Log}(1+z))$ be the principal branch of $(1+z)^\beta$, where β is a fixed complex number. Establish the validity for $z \in D_1(0)$ of the binomial expression

$$\begin{aligned}(1+z)^\beta &= 1 + \beta z + \frac{\beta(\beta-1)}{2!}z^2 + \frac{\beta(\beta-1)(\beta-3)}{3!}z^3 + \ldots \\ &= 1 + \sum_{n=1}^{\infty} \frac{\beta(\beta-1)\cdots(\beta-n+1)}{n!}z^n.\end{aligned}$$

(*Hint:* Compute derivatives of $(1+z)^\beta$.)
Comment. In this sense, the usual binomial formula holds for arbitrary, not just positive integer, β:

$$(1+z)^\beta = \sum_{n=0}^{\infty} \binom{\beta}{n} z^n.$$